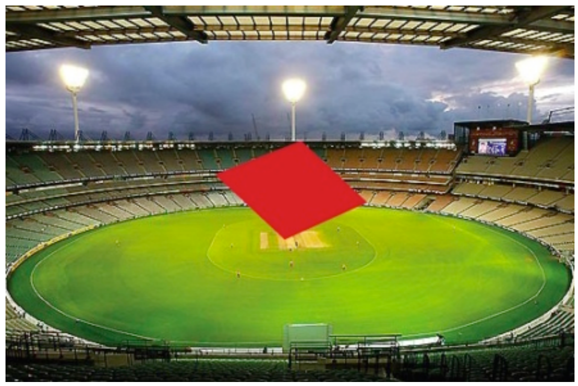
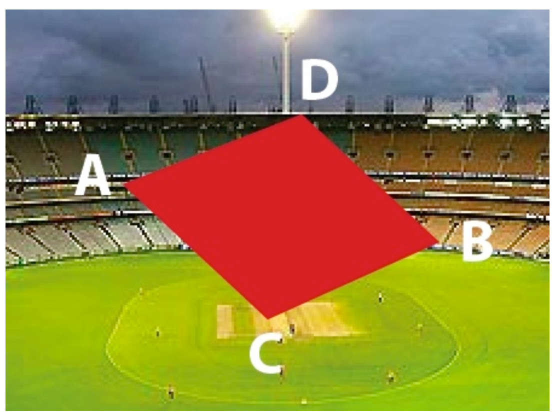
SUMMER MATHS QUIZ SOLUTIONS – PART 3

HARD 1

A mysterious square has materialized in the middle of the MCG, hovering in mid-air. The heights above the ground of three of its corners are 13, 21 and 34 meters. The fourth corner is higher still. How high?



Solution: The height of the fourth corner is 42 meters, as readers of *The Hitchhikers Guide to the Galaxy* would have already guessed.



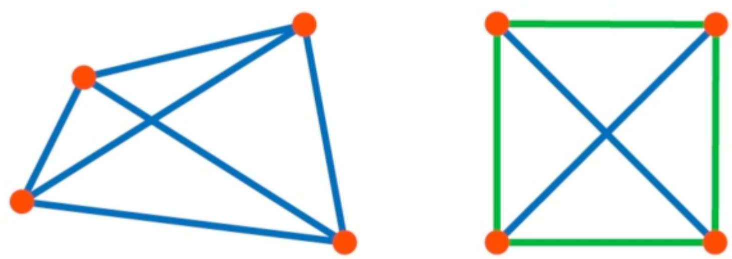
Label the corners A, B, C and D, as in the diagram above. Note that the rise from B to D equals the rise from C to A. (To see this, all that matters is that our shape is a parallelogram). So, using the letters to stand for the heights of the corners,

$$D - C = (D - B) + (B - C) = (A - C) + (B - C).$$

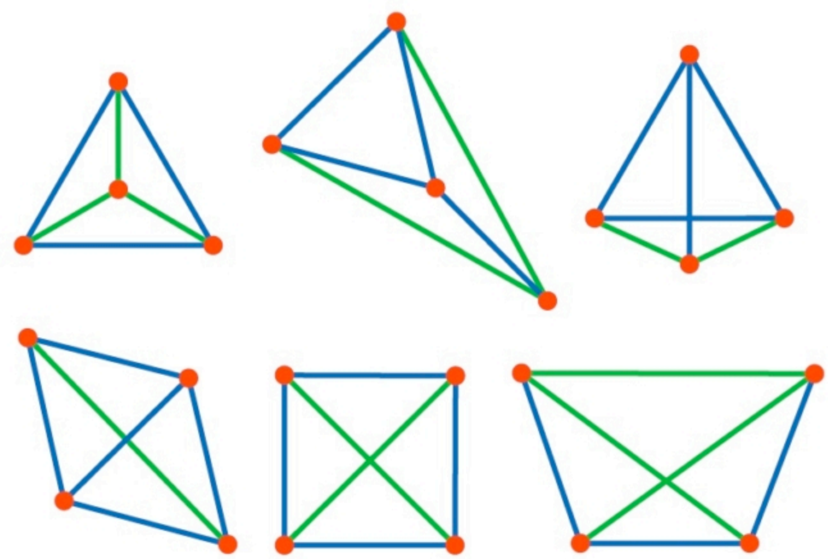
Using the values of A, B, and C, we find $D = 42$ meters.

HARD 2

If you choose four points in the plane, there are six distances between the various pairs of points. However, for a square there are actually only two distinct distances. Can you find another way to choose the points so there are only two different distances? And another?



Answer: There are six possible configurations in total:

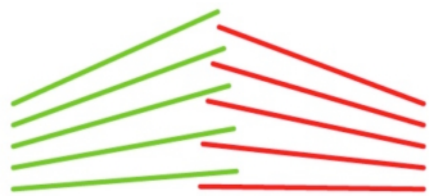


It's a bit tricky to see how to go methodically through the possibilities. The easiest approach seems to be to note that given there are six edges and two lengths, we can first choose three edges which have the same length.

There are then three scenarios, depending upon how those three equal edges meet: they may form an equilateral triangle (first four configurations); or three spokes around a vertex (first four configurations); or a connected path which doesn't close up (last two configurations). In each scenario, a little pondering makes clear that the only possibilities are those diagrammed.

HARD 3

A riffle shuffle of a deck of cards works as follows: the top half and the bottom half of the deck are separated, and then the cards are interleaved as shown in the diagram.



Suppose you start with the deck arranged in a certain order, and you keep riffle shuffling, always with the bottom half of the deck (the red cards in the diagram) contributing the lower cards in the riffle. How do you know that the deck will eventually return to the starting arrangement? If you use a deck of 10 cards, how many shuffles does it take to get back to the beginning?

Solution: If we start with 10 cards, then six shuffles are needed. To see this, we just keep careful track of the effect of the shuffles. Suppose we start with cards labeled 0123456789 from top to bottom. Then the results of the shuffles are as follows:

1st shuffle: 0516273849
2nd shuffle: 0753186429
3rd shuffle: 0876543219
4th shuffle: 0483726159
5th shuffle: 0246813579
6th shuffle: 0123456789

What about a deck of cards of some other size? Here's the argument that we will always eventually shuffle back to the starting arrangement.

Since there are only finitely many arrangements of the cards, it is clear that enough shuffles must result in some arrangement being repeated. So suppose N shuffles and M shuffles result in equal arrangements, with $N > M$. But then the arrangements before those shuffles must also have been the same. That is, $N–1$ shuffles and $M–1$ shuffles must also have resulted in equal arrangements. Similarly, $N–2$ and $M–2$ shuffles must have given the same arrangements, and so on. Counting back, we conclude that 0 shuffles, giving the starting arrangement, must be the same as the arrangement after $N–M$ shuffles.

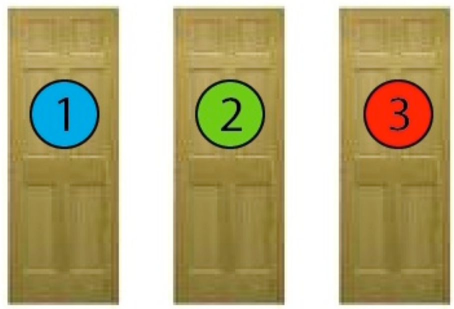
HARD 4

You have a pear and an apple. You're not actually hungry, so you pass the time seeing how a loop of wire can lie snugly along the surface of the apple. Must one of those wire loops, without any further bending, also lie snugly on the pear?

Solution: Imagine that the fruit are ghosts, and begin to pass them through one another. Draw along the lines where the skins of the fruits intersect. Those lines will consist of one or more loops: any such loop will do.

HARD 5

You appear on a TV game show, where you are given the choice of three doors. Behind one door is a brand new Ferrari, and behind the other two doors are man-eating Bengal tigers (cousins of QED cat). You pick Door 1, and your host, Simon the Likeable, opens Door 2, revealing a tiger. Beating away the tiger, Simon makes you an offer: "If you wish, you may switch your choice from Door 1 to Door 3". Should you?



Solution: This is the famous Monty Hall Problem. Tons have been written about this problem, including three separate books devoted to it. There are wrinkles to the solution. But basically, assuming you want to win the Ferrari (we'd actually prefer one of the tigers!), you should switch doors.

The odds depend subtly upon how the host Simon decides to open a door, and which door. We'll assume that Simon knows where the Ferrari is, will always open a door, and will never reveal the Ferrari. These assumptions, though often not spelled out explicitly, make sense for a real game show.

What if you stay with Door 1? Before Simon opens Door 2, the chances that you've got the Ferrari are 1/3. Simon then opens a door, revealing a tiger. *But*, he was *always* going to do that. Since Simon was always going to open a door to a tiger, his opening Door 2 has given you absolutely no better clue to whether the Ferrari is behind Door 1. Since you have new information about Door 1, your chances are still 1/3.

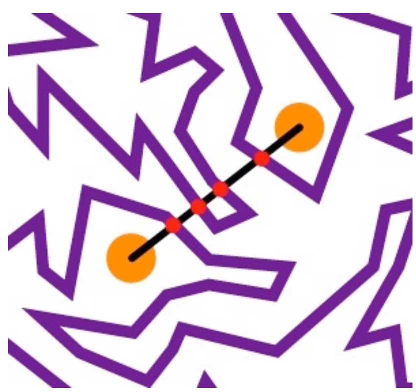
What if you plan to switch doors? Before Simon opens a door, you'll get the Ferrari if it is behind *either* Door 2 or Door 3, which is a 2/3 chance. The same is true after Simon opens the door: again, there is no new information, and your chances are still 2/3.

HARD 6

The weird purple curve below is actually a loop that never runs into itself. Part of the curve is hidden under the blue square. Once the square is removed, will it be possible to draw a path that connects the two orange points without running into the purple curve?

Solution: Yes, a path between the dots is possible.

Because the purple curve is a loop and doesn't run into itself, the loop has an inside and an outside. As long as the dots are both inside or both outside, we can draw a path between the dots.



How can we tell whether the dots are on the same side? Draw a straight line connecting the two dots. Each time this line crosses the loop, we go from inside to outside, or vice versa. The line crosses four times, and so both points are on the same side, and we can draw a path between them

By the way, this inside-outside stuff is intuitive, but is in fact incredibly subtle. Google "Jordan Curve Theorem" for more information.

HARD 7

As a magical Christmas present your father gives you infinitely many boxes, labeled 1, 2, 3, 4 and so on. Your mother's present to you is a set of "wooden fractions": one block in the shape of each fraction between 0 and 1. Unfortunately each box is only big enough to hold one fraction. Do you have enough boxes for all the fractions?

Solution: Surprisingly, you do have enough boxes. Suppose we arrange all the fractions between 0 and 1 in an infinite triangle, as indicated below:

1/2
1/3, 2/3
1/4, 2/4, 3/4
1/5, 2/5, 3/5, 4/5
1/6, 2/6, 3/6, 4/6, 5/6
Etc.

Now we just go along row by row, placing the fractions in the boxes. So, 1/2 goes in Box 1, and 1/3 goes in Box 2, and 2/3 goes in Box 3, and 1/4 goes in Box 4, and 2/4 goes in Box 5, and so on. Clearly for any given fraction, we'll eventually get to its row, and so that fraction will definitely be assigned a box.

Note that we're treating equal fractions as different wooden blocks: for example 1/2 and 2/4 each get their own separate box. If you prefer, you can ignore the reducible fractions: just leave those boxes empty, or move the other fractions up accordingly. Also, we've not assigned boxes for 0 or 1: if you wish, just put them at the top of the triangle, so they'll be assigned Boxes 1 and 2, respectively.

For more on this problem, Google "countability".

HARD 8

The Maths Masters have invited you to play a game. The game uses three dice, with sides numbered as follows:

DIE *A*: 3, 3, 5, 5, 7, 7,

DIE *B*: 2, 2, 4, 4, 9, 9,

DIE *C*: 1, 1, 6, 6, 8, 8.

To play the game, you each choose a die and roll it, with the high number scoring a point. You then put the dice back, each choose and roll again, and so on. First to 10 points wins. The Maths Masters seem to win much more often! Why? What is our Maths Masterly strategy?

Solution: We let you choose the die first. If you choose A, we choose B. If you choose B, we choose C, and if you choose C, we choose A. In each case, it turns out that we've got the better die. We have a 5/9 chance of winning on a given roll, and about a 69% chance of being the first to 10.

To see how this works, consider the following diagram showing the outcome table for die A versus die B. As you can see A has 5 chances out of 9 of winning.

B vs. A	3	5	7
2	A	A	A
4	B	A	A
9	B	B	B

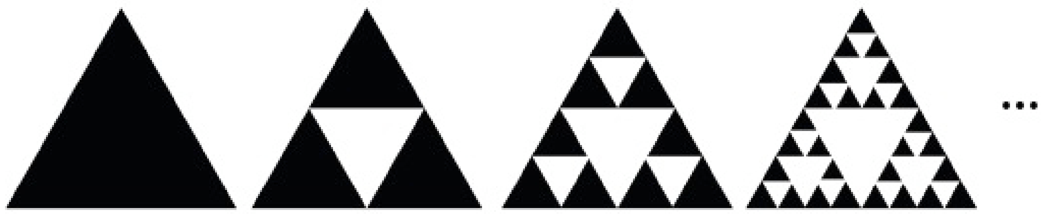
Similarly, it is easy to see that B has the odds against C, and, amazingly, C has the odds against A.

What this means is that the basic setup of the game is the same as Rock-Paper-Scissors. Of course in R-P-S, you can't usually get away with waiting for the other person to go first! But here, the unsuspecting customer may not realise how the dice are working.

The existence of such paradoxical dice is related to famous voting paradoxes, which we wrote about here.

HARD 9

Start with an equilateral triangle of area 1. Remove the middle equilateral triangle as indicated, leaving three smaller triangles. Then remove the middle triangles from each of those, and so on. If you keep on going forever, how much will be left of the original triangle? Anything?



Solution: There is no area left, but lots of points are left.

At the first step, we think of the original triangle as made of four equal smaller triangles. So, each smaller triangle has 1/4 the area of the original. Removing the middle one, we have left 3/4 of the original area.

At the next step, we perform the same operation on each of the three small triangles, again leaving 3/4 of the area of each smaller triangle. So, that leaves only 3/4 x 3/4 = 9/16 of the original area.

At each stage, we're only left with 3/4 of the area at the previous stage. After the *N*th removal, this leaves only $(3/4)^N$ of the original area, which is tiny. At the very end, there is no area left at all.

Still, there are lots of points left over. For example, consider any corner of any little black triangle you create on the way down: we think of the black triangles as including their edges and corners. Then, that vertex will be there at the very end. So, that's infinitely many points left in the end. (This is true even we think of the black triangles as not including their edges, but then it is much harder to see). In fact, there'll be many more points than these vertices: more even than will fit in the boxes of Hard Problem 7!

This strange creature is known as the Sierpinski gasket. There are many variations of it. In fact, if you are more careful with the sizes of the removed triangles, you can actually arrange to have some definite area left at the end.

HARD 10

What, if anything, is this crazy fraction?

$$1 + \cfrac{3 + \cfrac{1+\frac{3+\dots}{2+\dots}}{3+\frac{1+\dots}{3+\dots}}}{2 + \cfrac{2+\frac{2+\dots}{1+\dots}}{1+\frac{3+\dots}{2+\dots}}}$$

Solution: It's a fair amount of work to show that all those dots make sense. We'll assume here that the dots do make sense, and then show that it all sums to 7/4.

Let's write *M* for the number we're after. So,

$$M = 1 + \frac{3+\frac{M}{3+\frac{M}{3+\dots}}}{2+\frac{2+\frac{2+\dots}{M}}{M}}$$

Let's write this in the form

$$\boxed{M = 1 + \frac{K}{L}}$$

So, here K and L stand for

$$K = 3 + \frac{M}{3 + \frac{M}{3+\dots}}$$

and

$$L = 2 + \frac{2 + \frac{2+\dots}{M}}{M}$$

Then, the repeated nature of K and L, show that.

$$\boxed{\boldsymbol{K = 3 + \frac{M}{K}}}$$

and

$$\boxed{\boldsymbol{L = 2 + \frac{L}{M}}}$$

Now, together from the boxed equations, we can easily solve for M, K and L. Multiplying through by the denominators to eliminate the fractions, we have

$$LM = L + K, \quad K^2 = 3K + M, \quad LM = 2M + L.$$

From the first and third equations we see $K = 2M$. Plugging this into the second equation, we get $4M^2 = 7M$. Since $M = 0$ is clearly impossible (since $M = 1$ + something positive), we're left with the solution $M = 7/4$. (We can then also solve to get $K = 7/2$, and $L = 14/3$).